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Darboux transformations on timelike constant mean curvature surfaces

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Abstract

We give loop group theoretic reformulated Bäcklund transformations on constant mean curvature timelike surfaces in Minkowski 3-space. Further we present 1-soliton surfaces explicitly.

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Dedicated to Professor Hideki Omori on his sixtieth birthday

0. Introduction

In 19th century geometry, one of the central topics was the transformation theory of surfaces. The best known example might be the Bäcklund transformation on constant negative curvature (CNC) surfaces in Euclidean 3-space $\mathbf{E}^3$.

Originally Bäcklund transformation was defined as a transformation between CNC surfaces. More precisely, the Bäcklund transforamation was defined as a *line congruence* with pseudo-spherical properties. See Eisenhart [11], Palais and Terng [22] for more details.

Since the Gauss–Codazzi equations of a CNC surface become the Sine–Gordon equation with respect to the Chebyshev asymptotic coordinates, each Bäcklund transformation induces a transformation between Sine–Gordon fields. The induced transfromation is also called a Bäcklund transformation. A Bäcklund transformation is reffered as an operation of *adding solitions*. The permutability theorem of Bäcklund transformation due to L. Bianchi is interpreted as a *nonlinear superposition formula* for the Sine–Gordon fields.

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Bianchi also studied Bäcklund type transformation on constant positive curvature (CPC) surfaces [11]. In 1973, Ablowitz et al. introduced 2×2 matrix zero curvature representation of the Sine–Gordon equation [1].

Nowadays transformations for solutions to a soliton equation which *add solitons* to a given solution are generally called *Bäcklund transformations*. The construction of multi-soliton solutions (or quasiperiodic solutions) from the vacum solution by using Bäcklund transformation are called *the direct method*. Date [8] and Zakharov et al. [38] developed the direct method for solving the Sine–Gordon and related equations.

Recently, Sterling and Wente [30] studied Bäcklund transformation on constant mean curvature (CMC) surfaces. Their starting point is that each CMC surface corresponds to a CPC surface (Bonnet transform or parallel surface). They formulated a Bäcklund transformation for CMC surfaces by using the results due to Bianchi and Bonnet in modern literature. Further, Muto [26] has obtained another formulation of Bäcklund transformation on CMC surfaces. Muto reformulated the direct method by Date as *transformations on framings*. Muto's transformation on framings is a *Darboux form* of the Bäcklund transformation in the sense of Gu [14]. We shall call Muto's transformation *the Darboux transformation* on CMC surfaces (or equivalently, on Sinh–Laplace fields). We refer to [14] and Matveev and Salle [23] for general theory of Darboux transformations. Note that Sasaki has obtained the Darboux form of the Bäcklund transformations on Sine–Gordon fields [29]. Conformal geometric setting (or quaternionic calculus) for the Bäcklund transformations on CMC surfaces in Euclidean 3-space is established by Hertrich-Jeromin and Pedit [16]. In particular they showed that Bianchi–Bäcklund transformations are Ribaucoure sphere congruences.

The Sine–Gordon equation and the Sinh–Laplace equation are real form of the *complexified Sine–Gordon equation*. There is another interesting real form, *Sinh–Gordon equation*:

$$\omega_{uv} + \sinh \omega = 0.$$

Sinh–Gordon fields describe timelike surfaces of constant positive curvature (TCPC surfaces) and timelike surfaces of constant mean curvature (TCMC surfaces) in Minkowski 3-space.

As we saw in the previous paper [18], timelike surface of constant mean curvature admits 2×2 matrix formulation. It seems to be interesting to study Darboux transformations on TCMC surfaces with real distinct principal curvatures. Since the metric of a timelike surface is indefinite, Bäcklund and Darboux transformations are more complicated than those of CMC surfaces in Euclidean 3-space. In fact we can see in Section 2, there exist four kinds of Bäcklund transformations on TCMC surfaces with real distinct principal curvatures. In this paper we shall formulate Darboux transformations on TCMC surfaces and present 1-soliton surfaces explicitly.

1. Extended framings

In this paper we shall consider timelike CMC-1 immersions from $\mathbf{R}^2$ into $\mathbf{E}^3_1$ with real distinct principal curvatures. To study such immersions via the Darboux transformation theory, we need 2 × 2 matrix-formalism of the Gauss–Coddazi equations. For such approach we refer to Bobenko [4] and Inoguchi [18]. We shall devote this section to find special extended framings.

Let $\mathbf{E}^3_1$ be a Minkowski 3-space with natural Lorentz metric $\langle\cdot,\cdot\rangle$. The metric $\langle\cdot,\cdot\rangle$ is expressed as $\langle\cdot,\cdot\rangle = -\,\mathrm{d}\xi_1^2 + \mathrm{d}\xi_2^2 + \mathrm{d}\xi_3^2$ in terms of the natural coordinate system.

Let M be a connected 2-manifold and $\varphi : M \to \mathbf{E}^3_1$ an immersion. The immersion φ is said to be *timelike* if the induced metric I of M is Lorentzian. Hereafter we may assume that M is an orientable timelike surface in $\mathbf{E}^3_1$ immersed by φ. The Lorentzian metric of a timelike surface M determines a conformal structure on M. We treat M as a Lorentz surface with respect to this conformal structure and φ as a conformal immersion. Our general reference on Lorentz surfaces is T. Weinstein [37].

On a timelike surface M, there exists a local coordinate system (x, y) such that

$$I = e^{\omega}(-\,\mathrm{d}x^2 + \mathrm{d}y^2). \tag{1.1}$$

Such local coordinate system (x, y) is called a *Lorentz isothermal coordinate system*. Let (u, v) be the *null coordinate system derived from* (x, y). Namely (u, v) are defined by $u = x + y$, $v = -x + y$. The first fundamental form I is written by (u, v) as follows:

$$I = e^{\omega}\,\mathrm{d}u\,\mathrm{d}v. \tag{1.2}$$

Partial derivatives of φ satisfy the following formulae.

$$\langle\varphi_u, \varphi_u\rangle = \langle\varphi_v, \varphi_v\rangle = 0,\ \langle\varphi_u, \varphi_v\rangle = \tfrac{1}{2}e^{\omega}. \tag{1.3}$$

Now, let N be a local unit normal vector field to M. The vector field N is spacelike since M is timelike. The vector fields φ_u, φ_v as well as the normal N define a moving frame along φ. The compatibility conditions (Gauss–Codazzi equations) of the moving frame equations have the following form:

$$\omega_{uv} + \tfrac{1}{2}H^2 e^{\omega} - 2QRe^{-\omega} = 0, \tag{G}$$

$$H_u = 2e^{-\omega}Q_v,\ H_v = 2e^{-\omega}R_u. \tag{C}$$

Here H is the mean curvature of M defined by $H = 2e^{-\omega}\langle\varphi_{uv}, N\rangle$. The functions $Q := \langle\varphi_{uu}, N\rangle$ and $R := \langle\varphi_{vv}, N\rangle$ define global null 2-differentials $Q^{\#} := Q\,\mathrm{d}u^2$ and $R^{\#} := R\,\mathrm{d}v^2$ on M. We shall call the pair of differential $Q^{\#}$ and $R^{\#}$, the *Hopf pair of M*.

Next, we shall define the Gauss map of a timelike surface. Let M be a timelike surface and N a local unit normal vector field to M. For each $p \in M$ the point $\psi(p)$ of $\mathbf{E}^3_1$ canonically corresponding to the vector N_p lies in a unit pseudo 2-sphere since N is spacelike. The resulting smooth mapping $\psi : M \to S^2_1$ is called the *Gauss map* of M.

The constancy of mean or Gaussian curvature is characterised by the harmonicity of the Gauss map (see [4] and [25]).

Proposition 1.1. *The Gauss map of a timelike surface is harmonic if and only if the mean curvature is constant.*

Proposition 1.2. *Let M be a timelike surface. Assume that the Gaussian curvature K is nowhere zero on M and has a constant sign. Then the second fundamental form II gives M another (semi-) Riemannian metric. With respect to this metric II, the Gauss map of M is harmonic if and only if K is constant.*

On such surfaces special local coordinates are available [37, p. 213]. (For Euclidean case, see [4,11,30]).

Proposition 1.3. *Let M be a timelike surface of constant negative curvature -1. Then there exists a local coordinate system (x, y) around an arbitrary point of M such that:*

$$I = -\sin^2\frac{\phi}{2}\,\mathrm{d}x^2 + \cos^2\frac{\phi}{2}\,\mathrm{d}y^2, \quad II = \sin\frac{\phi}{2}\cos\frac{\phi}{2}(\mathrm{d}x^2 + \mathrm{d}y^2). \tag{1.4}$$

With respect to this coordinate system, the Gauss–Codazzi equation of the surface is written as following form (Sine–Laplace equation):

$$\phi_{xx} + \phi_{yy} = \sin\phi. \tag{1.5}$$

The above local coordinate system (x, y) is called a *second isothermic coordinate system.* Note that the hypothesis that M is free of umbilics in [32, Proposition 2] is superfluous. Since the Gaussian curvature K is constant -1, M has real distinct principal curvatures everywhere.

Proposition 1.4. *Let M be a timelike surface of constant curvature 4 with real distinct real two principal curvatures.*

(1) *If the principal vector corresponds to smaller principal curvature is timelike everywhere on M then there exists a local coordinate system (u, v) around any point of M such that*

$$I = \tfrac{1}{4}(\mathrm{d}u^2 - 2\cosh\omega\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}v^2),$$
$$II = \sinh\omega\,\mathrm{d}u\,\mathrm{d}v.$$

(2) *If the principal vector correspond to smaller principal curvature is spacelike everywhere on M then there exists a local coordinate system (u, v) around any point of M such that*

$$I = \tfrac{1}{4}(\mathrm{d}u^2 + 2\cosh\omega\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}v^2),$$
$$II = \sinh\omega\,\mathrm{d}u\,\mathrm{d}v.$$

With respect to this cordinate system, the Gauss–Codazzi equation of the surface is written as following form:

$$\omega_{uv} + \sinh\omega = 0.$$

The local coordinate system (u, v) in Proposition 1.4 is called a *Chebyshev-null coordinate system.* We shall call a timelike surface which satisfies the assumption (1) [resp. (2)] in Proposition 1.4 a timelike surface of (TS) ([resp. (ST)]) *type.*

Proposition 1.5. *Let M be a timelike surface of constant mean curvature 1. Assume that M has real distinct two principal curvatures.*

(1) *If M is of (TS) type then there exists a local coordinate system (x, y) around arbitrary point of M such that*

$$I = e^{\omega}(-\mathrm{d}x^2 + \mathrm{d}y^2),$$
$$II = 2e^{\omega/2}\left(-\sinh\frac{\omega}{2}\,\mathrm{d}x^2 + \cosh\frac{\omega}{2}\,\mathrm{d}y^2\right), \tag{1.6}$$
$$Q = R = 1/2.$$

(2) *If M is of (ST) type then there exists a local coordinate system (x, y) around arbitrary point of M such that*

$$I = e^{\omega}(-\mathrm{d}x^2 + \mathrm{d}y^2),$$
$$II = 2e^{\omega/2}\left(-\cosh\frac{\omega}{2}\,\mathrm{d}x^2 + \sinh\frac{\omega}{2}\,\mathrm{d}y^2\right), \tag{1.7}$$
$$Q = R = -1/2.$$

With respect to this coordinate system, the Gauss–Codazzi equation of the surface is written as following form:

$$-\omega_{xx} + \omega_{yy} + \sinh\omega = 0.$$

The local coordinate system (x, y) in the preceeding proposition is called an *isothermic coordinate system* or *isothermal curvature-line coordinate system* (cf. [4] and [16 (58)]).

Now we shall start the split-quaternioic representation. (see also [4] for Euclidean case). Let us denote the algebra of split-quaternions by $\mathbf{H}'$ and its natural basis by $\{\mathbf{1}, \mathbf{i}, \mathbf{j}', \mathbf{k}'\}$. The multiplication of $\mathbf{H}'$ is defined as follows:

$$\mathbf{i}\mathbf{j}' = -\mathbf{j}'\mathbf{i} = \mathbf{k}', \mathbf{j}'\mathbf{k}' = -\mathbf{k}'\mathbf{j}' = -\mathbf{i}, \mathbf{k}'\mathbf{i} = \mathbf{i}\mathbf{k}' = \mathbf{j}', \quad \mathbf{i}^2 = -\mathbf{1}, \mathbf{j}'^2 = \mathbf{k}'^2 = \mathbf{1}. \tag{1.8}$$

An element of $\mathbf{H}'$ is called a *split-quaternion*. For a split-quaternion $\xi = \xi_0\mathbf{1} + \xi_1\mathbf{i} + \xi_2\mathbf{j}' + \xi_3\mathbf{k}'$, the *conjugate* $\bar{\xi}$ of ξ is defined by

$$\bar{\xi} = \xi_0\mathbf{1} - \xi_1\mathbf{i} - \xi_2\mathbf{j}' - \xi_3\mathbf{k}'.$$

It is easy to see that $-\xi\bar{\xi} = -\xi_0^2 - \xi_1^2 + \xi_2^2 + \xi_3^2$. Hereafter we identify $\mathbf{H}'$ with a semi-Euclidean space $\mathbf{E}^4_2$:

$$\mathbf{E}^4_2 = (\mathbf{R}^4(\xi_0, \xi_1, \xi_2, \xi_3), -\mathrm{d}\xi_0^2 - \mathrm{d}\xi_1^2 + \mathrm{d}\xi_2^2 + \mathrm{d}\xi_3^2).$$

Let $G = \{\xi \in \mathbf{H}' | \xi\bar{\xi} = 1\}$ be the multiplication group of timelike unit split-quaternions. The Lie algebra $\mathfrak{g}$ of G is the imaginary part of $\mathbf{H}'$, that is,

$$\mathfrak{g} = \mathrm{Im}\mathbf{H}' = \{\xi_1\mathbf{i} + \xi_2\mathbf{j}' + \xi_3\mathbf{k}' | \xi_1, \xi_2, \xi_3 \in \mathbf{R}\}.$$

The Lie bracket of $\mathfrak{g}$ is simply the commutator of split-quaternion product. The Lie algebra $\mathfrak{g}$ is naturally identified with a Minkowski 3-space

$$\mathbf{E}^3_1 = (\mathbf{R}^3(\xi_1, \xi_2, \xi_3), -\mathrm{d}\xi_1^2 + \mathrm{d}\xi_2^2 + \mathrm{d}\xi_3^2)$$

as a metric linear space.

We shall introduce a 2×2 matricial expression of $\mathbf{H}'$ as follows:

$$\xi = \xi_0 \mathbf{1} + \xi_1 \mathbf{j}' + \xi_3 \mathbf{k}' \longleftrightarrow \begin{pmatrix} \xi_0 - \xi_3 & -\xi_1 + \xi_2 \\ \xi_1 + \xi_2 & \xi_0 + \xi_3 \end{pmatrix}. \tag{1.9}$$

In particular, the matricial expressions of the natural basis of $\mathbf{H}'$ are given by

$$\mathbf{1} \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} \longleftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$\mathbf{j}' \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{k}' \longleftrightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This correspondence gives an algebra isomorphism between $\mathbf{H}'$ and the algebra $\mathrm{M}_2\mathbf{R}$ of all matrices of degree 2. Under the identification (1.9), the group G of timelike unit split-quaternions corresponds to a special linear group:

$$\mathrm{SL}_2\mathbf{R} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2\mathbf{R} \,\middle|\, ad - bc = 1 \right\}.$$

The semi-Euclidean metric of $\mathbf{H}'$ corresponds to the following scalar product on $\mathrm{M}_2\mathbf{R}$.

$$\langle X, Y \rangle = \tfrac{1}{2}\{\mathrm{tr}(XY) - \mathrm{tr}(X)\mathrm{tr}(Y)\} \tag{1.10}$$

for all $X, Y \in \mathrm{M}_2\mathbf{R}$. The metric of G induced by (1.10) is a bi-invariant Lorentz metric of constant curvature -1. Hence the Lie group G is identified with an anti-de-Sitter 3-space H^3_1 of constant curvature -1 (see [7]).

The *vector product operation of* $\mathbf{E}^3_1$ is defined by

$$\xi \times \eta = (\xi_3\eta_2 - \xi_2\eta_3, \xi_3\eta_1 - \xi_1\eta_3, \xi_1\eta_2 - \xi_2\eta_1) \tag{1.11}$$

for $\xi = (\xi_1, \xi_2, \xi_3)$, $\eta = (\eta_1, \eta_2, \eta_3) \in \mathbf{E}^3_1$. The vector product $\xi \times \eta$ of ξ and η is written in terms of the Lie bracket as follows:

$$\xi \times \eta = \tfrac{1}{2}[\xi, \eta].$$

Now, we shall define the Hopf-fibering for a pseudo-sphere S^2_1. It is easy to see that the $\mathrm{Ad}(G)$-orbit of $\mathbf{k}' \in \mathfrak{g}$ is a pseudo-sphere:

$$S^2_1 = \{\xi \in \mathbf{E}^3_1 | \langle \xi, \xi \rangle = 1\}.$$

The Ad-action of G on S^2_1 is transitive and isometric. The isotropy subgroup of G at $\mathbf{k}'$ is $H^1_0 = \{\xi_0 \mathbf{1} + \xi_3 \mathbf{k}' | \xi_0^2 - \xi_3^2 = -1\}$. The group H^1_0 is a hyperbola in a Minkowski plane $\mathbf{E}^2_1(\xi_0, \xi_3)$. (This is a Lorentz analogue of $S^1 \subset \mathbf{E}^2(\xi_0, \xi_1)$). Note that the group H^1_0 is isomorphic to the multiplicative group $\mathbf{R}^* = \mathbf{R} \setminus \{0\}$.

The natural projection $\pi : G \to S^2_1$, given by $\pi(g) = \mathrm{Ad}(g)\mathbf{k}'$ for all $g \in G$, defines a principal H^1_0-bundle over S^2_1. We shall call this fibering the Hopf-fibering of S^2_1. Denote the isotropy subgroup H^1_0 at $\mathbf{k}'$ by K and its Lie algebra $\mathfrak{k} = \mathbf{R}\mathbf{k}'$. The tangent space of S^2_1 at the origin $\mathbf{k}'$ is given by $\mathfrak{m} = \mathbf{R}\mathbf{i} \oplus \mathbf{R}\mathbf{j}'$.

Let τ be an involution of $\mathfrak{g}$ defined by $\tau = \mathrm{Ad}(\mathbf{k}') = \prod_{\mathfrak{k}} - \prod_{\mathfrak{m}}$, where $\prod_{\mathfrak{k}}$ and $\prod_{\mathfrak{m}}$ are the projections from $\mathfrak{g}$ onto $\mathfrak{k}$ and $\mathfrak{m}$ respectively. The pair $(\mathfrak{g}, \tau)$ is a symmetric Lie algebra data for the semi-Riemannian symmetric space $S^2_1 = G/K$.

We shall rewrite the Gauss–Codazzi equations (G) and (C) in 2×2 matrix-form. Let $\varphi : M \to \mathbf{E}^3_1$ be a timelike surface with moving frame $(\varphi_u, \varphi_v, N)$. The local unit normal vector field N is given by

$$N = (\varphi_x \times \varphi_y)/|\varphi_x \times \varphi_y|.$$

We shall define a framing $\Phi^{[0]}$ by

$$\mathrm{Ad}(\Phi^{[0]})(\mathbf{i}, \mathbf{j}', \mathbf{k}') = (e^{-\omega/2}\varphi_x, e^{-\omega/2}\varphi_y, N), \det \Phi^{[0]} = e^{\omega/2}. \tag{1.12}$$

The $\mathbf{H}'$-valued function $\Phi^{[0]}$ satisfies the following system of linear differential equations (Choose $u_0 = \omega_u/4$ and $v_0 = \omega_v/4$ in [18 (2.6)]).

$$\frac{\partial}{\partial u}\Phi^{[0]} = \Phi^{[0]}U^{[0]}, \qquad \frac{\partial}{\partial v}\Phi^{[0]} = \Phi^{[0]}V^{[0]}, \tag{1.13}$$

$$U^{[0]} = \begin{pmatrix} 0 & -Qe^{-\omega/2} \\ \frac{H}{2}e^{\omega/2} & \frac{1}{2}\omega_u \end{pmatrix}, \qquad V^{[0]} = \begin{pmatrix} \frac{1}{2}\omega_v & -\frac{H}{2}e^{\omega/2} \\ Re^{-\omega/2} & 0 \end{pmatrix}. \tag{1.14}$$

In this section we concentrate our attention to TCMC surfaces of (ST) type or (TS) type.

Timelike circular cylinders and timelike hyperbolic cylinders are typical examples of timelike surfaces with properties (TS) and (ST) respectively.

Taking an isothermic coordinate system (x, y), we can normalise the Hopf pair so that $Q = R = -1/2$ [resp. $1/2$] on a timelike surface of (TS) [resp. (ST)] type with constant mean curvature 1. We get the following *zero curvature representation* for timelike surfaces with constant mean curvature 1 (cf. [4]).

Proposition 1.6 ([18]). *Let $\Phi^{[0]}_\lambda(u, v)$ be a solution of the following linear differential equations:*

$$\frac{\partial}{\partial u}\Phi^{[0]}_\lambda = \Phi^{[0]}_\lambda U^{[0]}(\lambda), \qquad \frac{\partial}{\partial v}\Phi^{[0]}_\lambda = \Phi^{[0]}_\lambda V^{[0]}(\lambda),$$

$$U^{[0]}(\lambda) = \frac{1}{2}\begin{pmatrix} 0 & \lambda e^{-\omega/2} \\ e^{\omega/2} & \omega_u \end{pmatrix}, \qquad V^{[0]}(\lambda) = \frac{1}{2}\begin{pmatrix} \omega_v & e^{\omega/2} \\ -\lambda^{-1}e^{-\omega/2} & 0 \end{pmatrix}. \tag{1.15}$$

Then

$$\varphi^{[0]}_\lambda = \frac{\partial}{\partial t}\Phi^{[0]}_\lambda \cdot (\Phi^{[0]}_\lambda)^{-1} - \psi^{[0]}_\lambda,$$
$$\psi^{[0]}_\lambda = \mathrm{Ad}(\Phi^{[0]}_\lambda)\mathbf{k}', \lambda = \pm e^{2t}, t \in \mathbf{R} \tag{1.16}$$

describes a loop of timelike constant mean curvature 1 surfaces with first fundamental form $I = e^{\omega}\, du\, dv$ and Hopf pair $Q_\lambda = -\lambda/2$, $R_\lambda = -\lambda^{-1}/2$. The Gauss mapping of each $\varphi^{[0]}_\lambda$ is $\psi^{[0]}_\lambda$. Here the function ω is a solution to the Sinh–Gordon equation (ShG).

Proposition 1.7. *Let $\Phi_\lambda^{[0]}(u, v)$ be a solution of the following linear differential equations:*

$$\frac{\partial}{\partial u}\Phi_\lambda^{[0]} = \Phi_\lambda^{[0]} U^{[0]}(\lambda), \qquad \frac{\partial}{\partial v}\Phi_\lambda^{[0]} = \Phi_\lambda^{[0]} V^{[0]}(\lambda),$$
$$U^{[10]}(\lambda) = \frac{1}{2}\begin{pmatrix} 0 & -\lambda e^{-\omega/2} \\ e^{\omega/2} & \omega_u \end{pmatrix}, \qquad V^{[0]}(\lambda) = \frac{1}{2}\begin{pmatrix} \omega_v & e^{\omega/2} \\ \lambda^{-1}e^{-\omega/2} & 0 \end{pmatrix}. \tag{1.17}$$

Then

$$\varphi_\lambda^{[0]} = \frac{\partial}{\partial t}\Phi_\lambda^{[0]} \cdot (\Phi_\lambda^{[0]})^{-1} - \psi_\lambda^{[0]}, \psi_\lambda^{[0]} = \mathrm{Ad}(\Phi_\lambda^{[0]})\mathbf{k}', \quad \lambda = \pm e^{2t}, t \in \mathbf{R} \tag{1.18}$$

describes a loop of timelike constant mean curvature 1 surfaces with first fundamental form $I = e^\omega \,\mathrm{d}u\,\mathrm{d}v$ and Hopf pair $Q_\lambda = \lambda/2$, $R_\lambda = \lambda^{-1}/2$. The Gauss mapping of each $\varphi_\lambda^{[0]}$ is $\psi_\lambda^{[0]}$. Here the function ω is a solution to the Sinh–Gordon equation (ShG).

We can see that each timelike surface $\varphi = \varphi_1^{[0]}$ of (TS) [resp. (ST)] type is associated to timelike surfaces $\varphi_\lambda^{[0]}$, $\lambda < 0$ of (ST) [resp. (TS)] type. Hereafter we shall restrict our attention to surfaces of (ST) type for simplicity. The description of (TS)-type surfaces is similar to that of (ST)-type surfaces.

For the study of Darboux transformations, We have to consider the following transform of the frame:

$$\Phi_\lambda := \Phi_{\lambda^2}^{[0]}\begin{pmatrix} \sqrt{\lambda}e^{-\omega/2} & 0 \\ 0 & 1/\sqrt{\lambda} \end{pmatrix}, \quad \lambda > 0. \tag{1.19}$$

The following formulae can be easily verified.

$$U(\lambda) := (\Phi_\lambda)^{-1}\frac{\partial}{\partial u}\Phi_\lambda = \frac{1}{2}\begin{pmatrix} -\omega_u & \lambda \\ \lambda & \omega_u \end{pmatrix},$$
$$V(\lambda) := (\Phi_\lambda)^{-1}\frac{\partial}{\partial v}\Phi_\lambda = \frac{1}{2}\begin{pmatrix} 0 & -\lambda^{-1}e^{\omega} \\ -\lambda^{-1}e^{-\omega} & 0 \end{pmatrix}. \tag{1.20}$$

Two matrix-valued functions (Lax pair) $U(\lambda)$ and $V(\lambda)$ are naturally extended for $\lambda \in \mathbf{R}^*$. Hereafter we shall denote the extended functions by the same letters $U(\lambda)$ and $V(\lambda)$.

Let $\Phi_\lambda : \mathbf{R}^2 \times \mathbf{R}^* \to \mathrm{SL}_2\mathbf{R}$ be a fundamental solution to the system:

$$\frac{\partial}{\partial u}\Phi_\lambda = \Phi_\lambda U(\lambda), \qquad \frac{\partial}{\partial v}\Phi_\lambda = \Phi_\lambda V(\lambda) \tag{1.21}$$

with initial condition $\Phi_\lambda(0, 0) \equiv \mathbf{1}$. We shall give a characterisation of the framing Φ_λ defined as above in terms of Lie algebra theory.

Recall the Hopf-fibering $\pi : \mathrm{SL}_2\mathbf{R} \to S_1^2$ of the pseudo 2-sphere S_1^2. The pseudo 2-sphere $S_1^2 = G/K = \mathrm{SL}_2\mathbf{R}/\mathbf{R}^*$ is represented as a Lorentzian symmetric space. The Lie algebra $\mathfrak{g}$ of G is decomposed as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Define the $\mathfrak{g}$-valued 1-form α_λ by $\alpha_\lambda = \Phi_\lambda^{-1}\,\mathrm{d}\Phi_\lambda$. Note that $\alpha_\lambda = U(\lambda)\,\mathrm{d}u + V(\lambda)\,\mathrm{d}v$. We can decompose $\alpha = \alpha_1$ along the decomposition

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ by $\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}}$. Further α has the type decomposition with respect to the conformal structure of M.

$$\alpha = \alpha\,\mathrm{d}u + \alpha''\,\mathrm{d}v, \quad \alpha' = \alpha'_{\mathfrak{k}} + \alpha'_{\mathfrak{m}}, \quad \alpha'' = \alpha''_{\mathfrak{k}} + \alpha''_{\mathfrak{m}}.$$

We can get the following proposition easily.

Proposition 1.8. *Let Φ_λ be a framing defined by* (1.21). *Then the $\mathfrak{g}$-valued 1-form α_λ by $\alpha_\lambda = \Phi_\lambda^{-1} d\Phi_\lambda^{-1}$ has the following form:*

$$\alpha_\lambda = \alpha_{\mathfrak{k}} + \lambda\alpha'_{\mathfrak{m}} + \lambda^{-1}\alpha''_{\mathfrak{m}}, \quad \alpha''_{\mathfrak{k}} = 0. \tag{1.22}$$

It is easy to see that Φ_λ is an extended framing (in the sense of harmonic map theory). With respect to the extended framing (1.21), the immersion formulae of timelike CMC surfaces are written in the following form.

Proposition 1.9. *Let Φ_λ be an extended framing* (1.21). *Then*

$$\varphi_\lambda = \frac{1}{2}\left\{\frac{\partial}{\partial t}\Phi_\lambda \cdot \Phi_\lambda^{-1} - \psi_\lambda\right\}, \qquad \psi_\lambda = \mathrm{Ad}(\Phi_\lambda)\mathbf{k}', \quad \lambda = \pm e^{2t}, t \in \mathbf{R} \tag{1.23}$$

describes a loop of timelike CMC-1 surfaces of (ST) type. Fundamental associated quantities of φ_λ are given as follows:

$$I_\lambda \equiv I = e^\omega\,\mathrm{d}u\,\mathrm{d}v,\ Q_\lambda = -\lambda^2/2,\ R_\lambda = -\lambda^{-2}/2. \tag{1.24}$$

The Gauss map of each φ_λ is ψ_λ. In particular (u, v) is an isothermal curvature-line coordinate system of $\varphi_{\pm 1}$.

By using the parallel surface procedure, we get the following immersion formula for timelike surfaces of constant curvature 4 (cf. [4]).

Corollary 1.10. *Let Φ_λ be an extended framing* (1.21). *Then*

$$F_\lambda = \frac{1}{2}\frac{\partial}{\partial t}\Phi_\lambda \cdot \Phi_\lambda^{-1} = \varphi_\lambda + \frac{1}{2}\psi_\lambda \tag{1.25}$$

describes a loop of timelike constant curvature 4 surfaces with real distinct principal curvatures. The Gauss map of each F_λ is ψ_λ. The fundamental associated quantities of F_λ are given as follows:

$$\begin{aligned} I_\lambda &= \tfrac{1}{4}(\lambda^2\,\mathrm{d}u^2 + 2\cosh\omega\,\mathrm{d}u\,\mathrm{d}v + \lambda^{-2}\,\mathrm{d}v^2), \\ II_\lambda &\equiv II = \sinh\omega\,\mathrm{d}u\,\mathrm{d}v. \end{aligned} \tag{1.26}$$

Example 1.11 (The vacuum solution). Here we shall solve the extended framing equation (1.21) explicitly for the vacuum solution $\omega \equiv 0$. The extended framing Φ_λ corresponding to the vacuum solution is given by

$$\Phi_\lambda = \begin{pmatrix} \cosh\{\frac{1}{2}(\lambda u - \lambda^{-1}v)\} & \sinh\{\frac{1}{2}(\lambda u - \lambda^{-1}v)\} \\ \sinh\{\frac{1}{2}(\lambda u - \lambda^{-1}v)\} & \cosh\{\frac{1}{2}(\lambda u - \lambda^{-1}v)\} \end{pmatrix} \tag{1.27}$$

We get the following formulae.

$$\varphi_\lambda = \frac{1}{2}\begin{bmatrix} \sinh(\lambda u - \lambda^{-1}v) \\ \lambda u + \lambda^{-1}v \\ -\cosh(\lambda u - \lambda^{-1}v) \end{bmatrix};$$

$$\psi_\lambda = \begin{bmatrix} -\sinh(\lambda u - \lambda^{-1}v) \\ 0 \\ \cosh(\lambda u - \lambda^{-1}v) \end{bmatrix};$$

$$I_\lambda \equiv I = \mathrm{d}u\,\mathrm{d}v, \quad H_\lambda \equiv 1, \quad Q_\lambda = -\lambda^2/2, \quad R_\lambda = -\lambda^{-2}/2. \tag{1.28}$$

Each timelike surface φ_λ is a connected component of a timelike hyperbolic cylinder: $\xi_1^2 + \xi_3^2 = 1/4$. (One sheet of the 2-sheeted hyperboloid determined by $\xi_3 < 0$). Note that the parallel surface $\hat{\varphi}_\lambda = \varphi_\lambda + \psi_\lambda$ is another connected component of the hyperbolic cylinder (another sheet determined by $\xi_3 > 0$). Each $\hat{\varphi}_\lambda$ has constant mean curvature 1 and Gauss map $-\psi_\lambda$. The parallel surface F_λ degenerates to a line (ξ_2-axis).

Remark. *In this paper we shall restrict our attention to TCMC surfaces with real distinct principal curvatures. We can also consider Bäcklund transformations on TCMC surfaces with real repeated principal curvatures. However such surfaces have restrictive shapes. More precisely we can prove the following proposition (cf.* [12] *and* [24 (4.80)]*).*

Proposition 1.12. *Let (M, φ) be a timelike surface of constant mean curvature. If (M, φ) has real repeated principal curvatures and corresponding eigenspaces are one-dimensional, then (M, φ) is locally congruent to a B-scroll.*

2. Bäcklund transformations

In her thesis [24], McNertney has developed the theory of Bäcklund transformations on timelike surfaces of constant positive curvature. In this section we shall recall and rewrite the results by McNertney for our use. In particular we shall show that, in timelike surface geometry, there are four kinds of Bäcklund transformations.

We start with the following definition.

Definition 2.1. Let M and $\tilde{M}$ be semi-Riemannian surfaces in Minkowski 3-space $\mathbf{E}_1^3$. A *line congruence* between M and $\tilde{M}$ is a local diffeomorphism $l : M \to \tilde{M}$ such that the line $\overrightarrow{p\tilde{p}}$, $\tilde{p} = l(p)$ is tangent to both M and $\tilde{M}$ for any p. The line congruence l is called a *Bäcklund transformation* if

(1) $\langle \overrightarrow{p\tilde{p}}, \overrightarrow{p\tilde{p}} \rangle$ is a constant independent of p;
(2) The scalar product $\langle N_p, \tilde{N}_{\tilde{p}} \rangle$ of two unit normal vectors N_p of M at p and $\tilde{N}_{\tilde{p}}$ of $\tilde{M}$ at $\tilde{p}$ is a constant independent of p.

The notion of Bäcklund transformations for timelike surfaces parallels the classical theory in so far as constant curvature is necessary for the existence of "pseudo-spherical line-congruence"; however, in timelike surface geometry, the Gaussian curvature should be *positive*. We can see that this leads to the investigation of various examples which have *no analogue* in Euclidean surface geometry. For example we shall see in Section 4, there exist TCMC surfaces corresponding to add-soliton solutions to (ShG).

Proposition 2.2 ([24]). *Let M and $\tilde{M}$ be timelike surfaces. Suppose a Bäcklund transformation exists between M and $\tilde{M}$.*
(1) *If $\langle \overrightarrow{p\tilde{p}}, \overrightarrow{p\tilde{p}} \rangle \equiv -r^2 < 0$ (timelike line congruence) then $\langle N_p, \tilde{N}_{\tilde{p}} \rangle = \cos\theta$ for some constant $\theta \in (0, \pi)$. In this case the Gaussian curvatures K of M and $\tilde{K}$ of $\tilde{M}$ are same positive constant $K = \tilde{K} \equiv \sin^2\theta/r^2$.*
(2) *If $\langle \overrightarrow{p\tilde{p}}, \overrightarrow{p\tilde{p}} \rangle \equiv r^2 > 0$ (spacelike line congruence) then $\langle N_p, \tilde{N}_{\tilde{p}} \rangle = \cosh\theta$ for some constant θ. In this case the Gaussian curvatures K of M and $\tilde{K}$ of $\tilde{M}$ are same constant $K = \tilde{K} \equiv \sinh^2\theta/r^2$.*

Proposition 2.3 ([Intergrability theorem 24]). *Let M be a timelike surface of constant positive curvature K.*
(1) *If $K = \sin^2\theta/r^2$ for some constants θ and r. Then for any initial data $(p_0; X_0) \in T_{p_0}M$ such that X_0 is a non principal timelike unit vector, there exists a timelike Bäcklund transformation l from a simply connected regin containing p_0 to a timelike surface $\tilde{M}$ with $\overrightarrow{p_0, \tilde{p}_0} = rX_0$ and $\langle N_p, \tilde{N}_{\tilde{p}} \rangle = \cos\theta$*
(2) *If $K = \sinh^2\theta/r^2$ for some constants θ and r. Then for any initial data $(p_0; X_0) \in T_{p_0}M$ such that X_0 is a non principal spacelike unit vector, there exists a spacelike Bäcklund transformation l from a simply connected region containing p_0 to a timelike surface $\tilde{M}$ with $\overrightarrow{p_0\tilde{p}_0} = rX_0$ and $\langle N_p, \tilde{N}_{\tilde{p}} \rangle = \cosh\theta$.*

Proposition 2.4. *Let M and $\tilde{M}$ be timelike constant positive curvature 4 surfaces of (ST) [resp. (TS)] type and let $l : M \to \tilde{M}$ a Bäcklund transformation. Then the Chebyshev-null coordinate system (u, v) of type (TS) [resp. (ST)] on M gives a Chebyshev-null coordinate system of type (TS) [resp. (ST)] on $\tilde{M}$ via l.*

The preceeding proposition implies that the Bäcklund transformation between timelike surface M and $\tilde{M}$ induce the transformations between Sinh–Gordon fields (cf. [16 (60)]) and (1.26) in [30]).

Corollary 2.5. *Let (M, F) be a timelike constant curvature 4 surface of type (TS). Then*
(1) *the timelike Bäcklund transform $\tilde{F}$ of F with data (r, θ) such that $\sin^2\theta/r^2 = 4$ is given by*

$$\tilde{F} = F + \frac{\sin\theta}{2}\left\{\frac{\cosh(\tilde{\omega}/2)}{\cosh(\omega/2)}F_x + \frac{\sinh(\tilde{\omega}/2)}{\sinh(\omega/2)}\right\}. \tag{2.1}$$

Here the coordinate system (x, y) *is defined by* $x = (u - v)/2$, $y = (u + v)/2$. *The Sinh–Gordon fields* ω *and* $\tilde{\omega}$ *are related by*

$$\frac{\partial}{\partial u}\left(\frac{\tilde{\omega} - \omega}{2}\right) = -\lambda \sinh\left(\frac{\tilde{\omega} + \omega}{2}\right),$$
$$\frac{\partial}{\partial v}\left(\frac{\tilde{\omega} + \omega}{2}\right) = \lambda^{-1} \sinh\left(\frac{\tilde{\omega} - \omega}{2}\right), \quad \lambda = \cot\theta + \csc\theta \in \mathbf{R}^*. \tag{TBs}$$

(2) *the spacelike Bäcklund transform* $\tilde{F}$ *of* F *with data* (r, θ) *such that* $\sinh^2\theta/r^2 = 4$ *is given by*

$$\tilde{F} = F + \frac{\sinh\theta}{2}\left\{\frac{\sinh(\tilde{\omega}/2)}{\cosh(\omega/2)}F_x + \frac{\cosh(\tilde{\omega}/2)}{\sinh(\omega/2)}F_y\right\}. \tag{2.2}$$

The Sinh–Gordon fields ω *and* $\tilde{\omega}$ *are related by*

$$\frac{\partial}{\partial u}\left(\frac{\tilde{\omega} + \omega}{2}\right) = -\lambda \cosh\left(\frac{\tilde{\omega} - \omega}{2}\right),$$
$$\frac{\partial}{\partial v}\left(\frac{\tilde{\omega} - \omega}{2}\right) = \lambda^{-1} \cosh\left(\frac{\tilde{\omega} + \omega}{2}\right), \quad \lambda = \coth\theta + \operatorname{csch}\theta \in \mathbf{R}^*. \tag{SBc}$$

Corollary 2.6. *Let* (M, F) *be a timelike constant curvature 4 surface of type (ST). Then*
(1) *the timelike Bäcklund transform* $\tilde{F}$ *of* F *with data* (r, θ) *such that* $\sin^2\theta/r^2 4$ *is given by*

$$\tilde{F} = F + \frac{\sin\theta}{2}\left\{\frac{\cosh(\tilde{\omega}/2)}{\sinh(\omega/2)}F_x + \frac{\sinh(\tilde{\omega}/2)}{\cosh(\omega/2)}F_y\right\}. \tag{2.3}$$

The Sinh–Gordon fields ω *and* $\tilde{\omega}$ *are related by*

$$\frac{\partial}{\partial u}\left(\frac{\tilde{\omega} + \omega}{2}\right) = -\lambda \cosh\left(\frac{\tilde{\omega} - \omega}{2}\right),$$
$$\frac{\partial}{\partial v}\left(\frac{\tilde{\omega} - \omega}{2}\right) = \lambda^{-1} \cosh\left(\frac{\tilde{\omega} + \omega}{2}\right), \quad \lambda = \cot\theta + \csc\theta \in \mathbf{R}^*. \tag{TBc}$$

(2) *the spacelike Bäcklund transform* $\tilde{F}$ *of* F *with data* (r, θ) *such that* $\sinh^2\theta/r^2 = 4$ *is given by*

$$\tilde{F} = F + \frac{\sinh\theta}{2}\left\{\frac{\sinh(\tilde{\omega}/2)}{\sinh(\omega/2)}F_x + \frac{\cosh(\tilde{\omega}/2)}{\cosh(\omega/2)}F_y\right\}. \tag{2.4}$$

The Sinh–Gordon fields ω *and* $\tilde{\omega}$ *are related by*

$$\frac{\partial}{\partial u}\left(\frac{\tilde{\omega} + \omega}{2}\right) = -\lambda \sinh\left(\frac{\tilde{\omega} - \omega}{2}\right),$$
$$\frac{\partial}{\partial v}\left(\frac{\tilde{\omega} - \omega}{2}\right) = \lambda^{-1} \sinh\left(\frac{\tilde{\omega} + \omega}{2}\right), \quad \lambda = \coth\theta + \operatorname{csch}\theta \in \mathbf{R}^*. \tag{SBs}$$

The parallel surface procedure induces transformations on timelike CMC surfaces. In the next section we shall treat induced transformations on timelike CMC surfaces and give a reformulation of induced (Bäcklund) transformations in terms of extended framings.

The formulae (TBs), (SBc), (TBc) and (SBc) are reffered as Bäcklund transformations of Sinh–Gordon fields. In Euclidean CMC surface geometry, Bäcklund transformations of Sinh–Laplace fields are introduced by Bianchi (cf. Theorem 5.4 in [31]).

In general Bäcklund transforms of a real Sinh–Laplace field is *complex* valued. Applying appropriate two successive Bäcklund transformations on a Sinh–Laplace field we get a new real Sinh–Laplace field (see Theorem 1.2 and Corollary 1.3 in [30]).

In our case – timelike CMC surfaces – we are working in *real* category, every Bäcklund transformation gives a new Sinh–Gordon fields for any given Sinh–Gordon fields.

Remark. *Babelon and Bernard* [1] *has been developed the soliton theory of (ShG). However they only considered (SBs).*

3. Bäcklund transformations via extended framings

In the preceding Section we saw that Bäcklund transformations between timelike surfaces induce transformations between Sinh–Gordon fields. Hereafter we shall only consider spacelike Bäcklund transformations (SBs) for simplicity.

In this section we shall reformulate (SBs) in terms of extended framings.

Let ω be a Sinh–Gordon field defined on the whole plane $\mathbf{R}^2$ and $\tilde{\omega}$ a Bäcklund transform of ω related by (SBs).

We shall define the function Γ by $\tilde{\omega} - \omega = 2\log\Gamma$. Then the function Γ satisfies the following Riccati type differential equations.

$$\frac{\partial\Gamma}{\partial u} = -\frac{\lambda}{2}\Gamma^2 - \omega_u\Gamma + \frac{\lambda}{2},$$
$$\frac{\partial\Gamma}{\partial v} = -\frac{\lambda^{-1}}{2}e^{\omega}\Gamma^2 - \frac{\lambda^{-1}}{2}e^{-\omega}. \tag{3.1}$$

As is well known, Riccati type differential equations can be linearised in the following way (cf. [29] and [35].)

Proposition 3.1. *Let ω and $\tilde{\omega}$ be Sinh–Gordon fields. Then $\tilde{\omega}$ is a Bäcklunk transform of ω related by (SBs) if and only if the pair of functions (f, g) such that $\tilde{\omega} - \omega = 2\log(f/g)$ is a solution to the following equations:*

$$\frac{\partial}{\partial u}(f, g) = (f, g)U(\lambda), \qquad \frac{\partial}{\partial v}(f, g) = (f, g)V(\lambda),$$
$$U(\lambda) = \frac{1}{2}\begin{pmatrix} -\omega_u & \lambda \\ \lambda & \omega_u \end{pmatrix}, \qquad V(\lambda) = \frac{1}{2}\begin{pmatrix} 0 & -\lambda^{-1}e^{\omega} \\ -\lambda^{-1}e^{-\omega} & 0 \end{pmatrix}. \tag{3.2}$$

Namely (f, g) is a row solution of the extended framing equation (1.21).

This proposition implies the following reformulation of (SBs) in terms of extended framings.

Corollary 3.2. *Let Φ_λ be an extended framing defined by* (1.21) *with Sinh–Gordon field ω and (f, g) a row vector of Φ_λ. Then $\tilde{\omega} := \omega + 2\log\Gamma$, $\Gamma - f/g$ is a Bäcklund transform of ω related by (SBs). Hence the timelike surface $\tilde{\varphi}$ corresponds to $\tilde{\omega}$ and timelike surface φ corresponds to ω have same conformal structures.*

The Riccati form (3.1) of (SBs) implies that the Sinh–Gordon fields have infinite numbers of conserved densities. In fact we get the following.

Corollary 3.3 (Conservation law). *The Sinh–Gordon field ω satisfies the following conservation law:*

$$\frac{\partial}{\partial u}(\lambda^{-1}e^{\omega}\Gamma) + \frac{\partial}{\partial v}(\lambda\Gamma + \omega_u) = 0.$$

Remark. *Hertrich-Jeromin and Pedit have obtained a quaternionic representation of the Riccati form for Sinh–Laplace fields in* [16, *Eq.* (62)].

4. Darboux transformations

In this section we shall reformulate Bäcklund transformations on timelike CMC surfaces as transformations on extended framings. More precisely, we shall consider the following problem:

Let Φ_λ be an extended framing (1.21). Find polynomial maps R_λ into the twisted loop group of $SL_2\mathbf{R}$ (or $GL_2\mathbf{R}$) such that $\Phi_\lambda R_\lambda$ is also a solution to the extended framing equation (1.21).

Such polynomial maps R_λ are traditionally called *Darboux matrices* [23]. The results in this section are Lorentzian version of those in [26].

We start with some preliminaries on loop groups. Let us denote τ the involution of a Lorentzian symmetric space $S^2_1 = SL_2\mathbf{R}/\mathbf{R}^*$ as before. More explicitly, τ is given by $\tau = \mathrm{Ad}(\mathbf{k}')$. Note that τ can be extended to general linear group $GL_2\mathbf{R}$ in a natural manner. We shall denote the extended involution by the same letter. We shall use the following loop groups.

The free loop group of $GL_2\mathbf{R}$;

$$LGL_2\mathbf{R} := \{\gamma : \mathbf{R}^* \to GL_2\mathbf{R} \mid \text{smooth}\};$$

The free loop group of $SL_2\mathbf{R}$;

$$LSL_2\mathbf{R} := \{\gamma : \mathbf{R}^* \to SL_2\mathbf{R} \mid \text{smooth}\};$$

The twisted loop group of $GL_2\mathbf{R}$;

$$LGL_2\mathbf{R}_\tau := \{\gamma \in LGL_2\mathbf{R} \mid \tau\gamma(\lambda) = \gamma(-\lambda)\};$$

The twisted positive power polynomial loop group of $\mathrm{GL}_2\mathbf{R}$;

$$L^+_{\mathrm{pol}}\mathrm{GL}_2\mathbf{R}_\tau := \left\{ \gamma \in L\mathrm{GL}_2\mathbf{R}_\tau \,\middle|\, \gamma(\lambda) = \sum_{j=0}^{N} \gamma_j \lambda^j \text{ for some } N \in \mathbf{N} \right\}.$$

The twisted loop group $L\mathrm{SL}_2\mathbf{R}_\tau$ and the twisted positive power polynomial loop group $L^+_{\mathrm{pol}}\mathrm{SL}_2\mathbf{R}_\tau$ of $\mathrm{SL}_2\mathbf{R}$ are defined in a similar fashion.

Note that every extended framing (1.21) can be regard as a mapping from $\mathbf{R}^2$ into $L\mathrm{SL}_2\mathbf{R}_\tau$.

Proposition 4.1. *Let Φ_λ be an extended framing defined by* (1.21) *with Sinh–Gordon field ω. Then $\tau\Phi_{-\lambda}$ is also an extended framing satisfyning* (1.21) *with Sinh–Gordon field ω.*

Let us denote p_{ij} by the (i, j)-entry of the $L\mathrm{GL}_2\mathbf{R}$-valued function P_λ. Then we get the following useful lemma.

Lemma 4.2. *Let P_λ be a $L\mathrm{GL}_2\mathbf{R}$-valued function. Then P_λ is τ-equivariant if and only if the diagonal entries p_{11} and p_{22} are even function of λ and the off diagonal entries p_{12} and p_{21} are odd functions of λ.*

Let Φ_λ be an extended framing defined by (1.21). We shall construct $L^+_{\mathrm{pol}}\mathrm{GL}_2\mathbf{R}_\tau$-valuled Darboux matrixes R_λ (not necessary $L^+_{\mathrm{pol}}\mathrm{SL}_2\mathbf{R}_\tau$-valued). By Lemma 4.2, R_λ has the following form:

$$R_\lambda = \begin{pmatrix} \Sigma f_{2j}\lambda^{2j} & \Sigma g_{2j+1}\lambda^{2j+1} \\ \Sigma f_{2j+1}\lambda^{2j+1} & \Sigma g_{2j}\lambda^{2j} \end{pmatrix}. \tag{4.1}$$

Without loss of generality, we can assume that the Darboux matrix $R_\lambda : \mathbf{R}^2 \to L^+_{\mathrm{pol}}\mathrm{GL}_2\mathbf{R}_\tau$ has the following form.

$$R_\lambda(u, v) = \sum_{j=0}^{N} R_j(u, v)\lambda^j,$$
$$R_N(u, v) = \begin{cases} \mathbf{1}, & \text{if } N = 2m, \\ \mathbf{j}', & \text{if } N = 2m + 1. \end{cases} \tag{4.2}$$

Now we shall state our existence results on Darboux matrices. We shall prepare a notational convensions. For any loop $P : \mathbf{R}^2 \to L\mathrm{GL}_2\mathbf{R}_\tau$, we shall define two functions p_1 and p_2 by $p_1(\lambda) = p_{11}(\lambda) + p_{21}(\lambda)$, $p_2(\lambda) = p_{12}(\lambda) + p_{22}(\lambda)$.

We shall take real vectors $\vec{\lambda}, \vec{c} \in \mathbf{R}^N$ such that $\lambda_i \neq 0$, $\lambda_i \neq \pm\lambda_j$ and $c_j \neq 0$. We shall call such a pair of vectors $(\vec{\lambda}, \vec{c})$ a *Darboux data*. for any Darboux data $(\vec{\lambda}, \vec{c})$, we shall define auxility matrices as follows:

$$\Lambda := diag(\lambda_1, \ldots, \lambda_N),$$
$$\vec{\Gamma}_k := (\phi_k(\lambda_1) + (-1)^k c_1 \phi_k(-\lambda_1), \ldots, \phi_k(\lambda_N) + (-1)^k c_N \phi_k(-\lambda_N))', \; k = 1, 2,$$

$$A_1 := \begin{cases} (\vec{\Gamma}_1, \Lambda\vec{\Gamma}_2, \ldots, \Lambda^{2m-2}\vec{\Gamma}_1,\ \Lambda^{2m-1}\vec{\Gamma}_2), & \text{if } N = 2m, \\ (\vec{\Gamma}_1, \Lambda\vec{\Gamma}_2, \ldots, \Lambda^{2m-1}\vec{\Gamma}_2,\ \Lambda^{2m}\vec{\Gamma}_1), & \text{if } N = 2m+1, \end{cases}$$

$$A_2 := \begin{cases} (\vec{\Gamma}_2, \Lambda\vec{\Gamma}_1, \ldots, \Lambda^{2m-2}\vec{\Gamma}_2,\ \Lambda^{2m-1}\vec{\Gamma}_1), & \text{if } N = 2m, \\ (\vec{\Gamma}_2, \Lambda\vec{\Gamma}_1, \ldots, \Lambda^{2m-1}\vec{\Gamma}_1,\ \Lambda^{2m}\vec{\Gamma}_1), & \text{if } N = 2m+1. \end{cases}$$

Lemma 4.3. *Let* $(\vec{\lambda}, \vec{c})$ *be a Darboux data for an extended framing* Φ_λ *defined by* (1.21) *with Sinh–Gordon field* ω. *If* $\det A_1 \neq 0$ *and* $\det A_2 \neq 0$ *on* $\mathbf{R}^2$ *then there exists a unique singular mapping* $R_\lambda : \mathbf{R}^2 \to L^+_{\mathrm{pol}}\mathrm{GL}_2 R_\tau$ *of the form* (4.2) *which satisfies the following adding-soliton properties:*

$$(1, -c_j)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Phi_\lambda R_\lambda|_{\lambda=\lambda_j}(0,0), \quad j = 1, \ldots, N. \tag{ASP}$$

Proof. The adding-soliton properties (ASP) determine uniquely the mapping R_λ. In fact, if we write R_λ as

$$R_\lambda = \begin{pmatrix} \Sigma f_{2j}\lambda^{2j} & \Sigma g_{2j+1}\lambda^{2j+1} \\ \Sigma f_{2j+1}\lambda^{2j+1} & \Sigma g_{2j}\lambda^{2j} \end{pmatrix}$$

then (ASP) are equivalent to the following linear algebraic equations for $\vec{f} = (f_0, \ldots, f_{N-1})^t$, $\vec{g} = (g_0, \ldots, g_{N-1})^t$:

(1) $N = 2m$:

$$A_1\vec{f} = -\Lambda^{2m}\vec{\Gamma}_1, \qquad A_2\vec{g} = -\Lambda^{2m}\vec{\Gamma}_2.$$

(2) $N = 2m+1$:

$$A_1\vec{f} = -\Lambda^{2m+1}\vec{\Gamma}_2, \qquad A_2\vec{g} = -\Lambda^{2m+1}\vec{\Gamma}_1.$$

Hence the results follows. □

Remark. *The adding-soliton properties (ASP) can be rewritten in the following form:*

$$\psi_i(\lambda_j) + (-1)^i c_j \psi_i(-\lambda_j) = 0, \quad i = 1, 2, \quad j = 1, \ldots, N$$

for $\Psi_\lambda = \Phi_\lambda R_\lambda$ (cf. [7, p. 134 (3.2)] and [8, p. 138 (22)].)

By using the Cramer's formula, we get

Lemma 4.4. *Let* R_λ *be a maping in Lemma* 2.3, *then*

$$f_0 = (-1)^N \prod_{i=1}^{N} \lambda_i \frac{\det A_2}{\det A_1}, \qquad g_0 = (-1)^N \prod_{i=1}^{N} \lambda_i \frac{\det A_1}{\det A_2}.$$

The following is the main result of this article which may be considered as a timelike version of Muto [26].

Theorem 4.5. *Let Φ_λ be an extended framing defined by* (1.21) *with Sinh–Gordon field ω. For any Darboux data $(\vec{\lambda}, \vec{c})$ such that* $\det A_1 \neq 0$ *and* $\det A_2 \neq 0$ *on* $\mathbf{R}^2$ *there exists unique singular Darboux matrix $R_\lambda : \mathbf{R}^2 \to L^+_{\mathrm{pol}}\mathrm{GL}_2 R_\tau$ such that*
(1) *R_λ has a form* (4.2);
(2) *R_λ satisfies (ASP);*
(3) *$\tilde{\omega} := \omega + 2\log(\det A_2/\det A_1)$ is an iterated Bäcklund transform of ω related by (SBs);*
(4) *$\tilde{\omega}_u = (-1)^N \omega_u - (f_{N-1} - g_{N-1})$;*
(5) *The determinant* $\det R_\lambda$ *of R_λ is independent of u and v.*
Further $\det R_\lambda$ *is a polynomial of λ of degree $2N$;*
(6) $\det R_\lambda = 0$ *at* $\lambda = \pm\lambda_j$, $j = 1, \ldots, N$.

Proof. Let R_λ be a singular mapping constructed in Lemma 4.3. We shall define auxility functions $L^{(1)}$, $L^{(2)}$, $M^{(1)}$ and $M^{(2)}$ by

$$L^{(1)}(u, v, ; \lambda) := \frac{\partial}{\partial u}\psi_1 - \frac{\lambda}{2}\psi_2 + \frac{(-1)^N}{2}\{\omega_u - (-1)^N (f_{N-1} - g_{N-1})\}\psi_1;$$
$$L^{(2)}(u, v, ; \lambda) := \frac{\partial}{\partial u}\psi_2 - \frac{\lambda}{2}\psi_1 - \frac{(-1)^N}{2}\{\omega_u - (-1)^N (f_{N-1} - g_{N-1})\}\psi_2;$$
$$M^{(1)}(u, v, ; \lambda) := \frac{\partial}{\partial v}\psi_1 + \frac{e^{-\omega}}{2}\lambda^{-1}\frac{f_0}{g_0}\psi_2;$$
$$M^{(2)}(u, v, ; \lambda) := \frac{\partial}{\partial v}\psi_2 + \frac{e^{-\omega}}{2}\lambda^{-1}\frac{g_0}{f_0}\psi_1.$$

Where ψ_1 and ψ_2 are defined by $\psi_1 = \psi_{11} + \psi_{21}$, $\psi_2 = \psi_{12} + \psi_{22}$ for $\Psi_\lambda = \Phi_\lambda R_\lambda$. One can show that these four functions vanish. For simplicity we shall only show that $L^{(1)} = 0$. The function $L^{(1)}$ has the following form:

$$L^{(1)}(u, v, \lambda) = \sum F_{2j}(u, v)\lambda^{2j}\phi_1 + \sum F_{2j+1}(u, v)\lambda^{2j+1}\phi_2.$$

In particular the degree of $L^{(1)}$ is $N - 1$. Evidently the auxility function $L^{(1)}$ satisfies the following condition (see Remark to Lemma 4.3.)

$$L^{(1)}(u, v; \lambda_j) - c_j L^{(1)}(u, v; -\lambda_j) = 0.$$

Hence the vector valued function $\vec{F} = (F_0, F_1, \ldots, F_{N-1})^t$ satisfies $A_1\vec{F} = \vec{0}$. Since A_1 is invertible we get $\vec{F} = \vec{0}$ and hence $L^{(1)} \equiv 0$. □

As we remarked in the end of Section 2, in Euclidean CMC surface geometry, we should have even soliton number $N = 2m$.

In timelike CMC surface geometry, we need not complexify the Lie group $\mathrm{SL}_2\mathbf{R}$ we can construct Darboux matrix of odd soliton number satisfying (ASP). Thus timelike surfaces corresponding to odd-soliton solution have no Euclidean analogues.

If we choose a seed extended framing Φ_λ as the extended framing corresponds to the vacuum solution in Example 1.11, then the Darboux transform $\tilde{\Phi}_\lambda = \Phi_\lambda R_\lambda/\sqrt{|\det R_\lambda|}$ gives multi-soliton solutions of the Sinh–Gordon equation.

Example 4.6 (One-soliton solutions). Let Φ_λ be an extended framing defined by (1.21) with Sinh–Gordon field ω and $(\lambda_1, c_1) = (k, c) \in \mathbf{R}^2$ a Darboux data for Φ_λ. Then the Darboux matrix R_λ of degree 1 determined by the data (k, c) is given by

$$R_\lambda = \lambda \mathbf{j}' + \begin{pmatrix} f_0 & 0 \\ 0 & g_0 \end{pmatrix};$$

$$f_0 = -k\frac{\phi_2(k) + c\phi_2(-k)}{\phi_1(k) - c\phi_1(-k)}, \qquad g_0 = -k\frac{\phi_1(k) - c\phi_1(-k)}{\phi_2(k) + c\phi_2(-k)}.$$

The Bäcklund transform $\tilde{\omega}$ of ω is

$$\tilde{\omega} = \omega + 2\log\frac{\phi_2(k) - c\phi_2(-k)}{\phi_1(k) + c\phi_1(-k)}.$$

Now we choose Φ_λ as an extended framing corresponding to the vacuum solution $\omega \equiv 0$ described in Example 1.11 and $c = 1$. Then the Darboux matrix determined by the data $(k, 1)$ is

$$R_\lambda = \begin{pmatrix} -k\coth\frac{1}{2}(ku - k^{-1}v) & \lambda \\ \lambda & -k\tanh\frac{1}{2}(ku - k^{-1}v) \end{pmatrix}.$$

This Darboux matrix R_λ is singular at $\lambda = \pm k$. The Bäcklund transform $\tilde{\omega}$ of the vacuum solution is given by

$$\tilde{\omega}(u, v) = 4\tanh^{-1}\exp(ku - k^{-1}v).$$

This solution $\tilde{\omega}$ is a one-soliton solution of the Sinh–Gordon equation (ShG).

The loop of timelike surfaces corresponds to the 1-soliton solution $\tilde{\omega}$ is

$$\tilde{\varphi}_\lambda = \begin{bmatrix} -\dfrac{\lambda k}{k^2 - \lambda^2}\tanh\dfrac{ku - k^{-1}v}{2}\cosh(\lambda u - \lambda^{-1}v) + \dfrac{1}{2}\dfrac{k^2 + \lambda^2}{k^2 - \lambda^2}\sinh(\lambda u - \lambda^{-1}v) \\ -\dfrac{\lambda k}{k^2 - \lambda^2}\tanh\dfrac{ku - k^{-1}v}{2} + \dfrac{1}{2}(\lambda u - \lambda^{-1}v) \\ \dfrac{\lambda k}{k^2 - \lambda^2}\tanh\dfrac{ku - k^{-1}v}{2}\sinh(\lambda u - \lambda^{-1}v) - \dfrac{1}{2}\dfrac{k^2 + \lambda^2}{k^2 - \lambda^2}\cosh(\lambda u - \lambda^{-1}v) \end{bmatrix}.$$

The Gauss map $\tilde{\psi}_\lambda$ of each $\tilde{\varphi}_\lambda$ is

$$\tilde{\psi}_\lambda = \frac{k^2 + \lambda^2}{k^2 - \lambda^2}\begin{bmatrix} -\sinh(\lambda u - \lambda^{-1}v) \\ 0 \\ \cosh(\lambda u - \lambda^{-1}v) \end{bmatrix} + \frac{-2\lambda k}{k^2 - \lambda^2}\begin{bmatrix} \coth(ku - k^{-1}v)\cosh(\lambda u - \lambda^{-1}v) \\ -\operatorname{cosech}(ku - k^{-1}v) \\ -\coth(ku - k^{-1}v)\sinh(\lambda u - \lambda^{-1}v) \end{bmatrix}.$$

The parallel surface $\tilde{F}_\lambda$ of $\tilde{\varphi}_\lambda$ with constant curvature 4 is

$$\tilde{F}_\lambda = \frac{\lambda k}{k^2 - \lambda^2}\begin{bmatrix} \operatorname{cosech}(ku - k^{-1}v)\cosh(\lambda u - \lambda^{-1}v) \\ -\coth(ku - k^{-1}v) \\ -\operatorname{cosech}(ku - k^{-1}v)\sinh(\lambda u - \lambda^{-1}v) \end{bmatrix} + \frac{1}{2}(\lambda u + \lambda^{-1}v)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

5. Bäcklund transformations on spacelike surfaces

In this paper we have considered Bäcklund transformations on timelike surfaces. In Minkowski 3-space $\mathbf{E}^3_1$, we can consider Bäcklund transformations on spacelike surfaces. In this section we shall give some fundamental results on this topic. We can deduce the following proposition in a similar way in [6,27].

Proposition 5.1. *Let (M, F) and $(\tilde{M}, \tilde{F})$ be spacelike surfaces. Suppose a Bäcklund transformation exists between (M, F) and $(\tilde{M}, \tilde{F})$. If $\langle \overrightarrow{p\tilde{p}}, \overrightarrow{p\tilde{p}} \rangle \equiv r^2 > 0$ then $\langle N_p, \tilde{N}_{\tilde{p}} \rangle = \cosh\theta$ for some constant θ. In addition the Gaussian curvatures K of M and $\tilde{K}$ of $\tilde{M}$ are same positive constant $K = \tilde{K} \equiv \sinh^2\theta/r^2$.*

We can consider Bäcklund transformations between spacelike surfaces and time like surfaces. The following proposition is due to Palmer [28].

Proposition 5.2 ([28]). *Let (M, F) be a spacelike surface and $(\tilde{M}, \tilde{F})$ be timelike surfaces. Suppose a Bäcklund transformation exists between M and $\tilde{M}$. If $\langle \overrightarrow{p\tilde{p}}, \overrightarrow{p\tilde{p}} \rangle \equiv r^2 > 0$ then $\langle N_p, \tilde{N}_{\tilde{p}} \rangle = \sinh\theta$ for some constant θ. In addition the Gaussian curvatures K of M and $\tilde{K}$ of $\tilde{M}$ are same negative constant $K = \tilde{K} \equiv -\cosh^2\theta/r^2$.*

The Bäcklund transformations between spacelike surfaces of $K \equiv -1$ and timelike surfaces of $K \equiv -1$ induce transformations between Sinh–Laplace fields and Sine–Laplace fields as follows [28] (cf. [32].).

Corollary 5.3. *Let (M, F) and $(\tilde{M}, \tilde{F})$ be surfaces as in the preceeding proposition and (x,y) be the second isothermic coordinate system of (M, F). Namely the first and second fundamental forms of (M, F) is written in the following forms:*

$$I = \sinh^2\frac{\omega}{2}\,\mathrm{d}x^2 + \cosh^2\frac{\omega}{2}\,\mathrm{d}y^2, \qquad II = \sinh\frac{\omega}{2}\cosh\frac{\omega}{2}(\mathrm{d}x^2 + \mathrm{d}y^2). \tag{5.1}$$

Then the coordinate system (x, y) is a second isothermic coordinate system of $(\tilde{M}, \tilde{F})$ described in Proposition 1.3. Namely, with respect to the coordinates (x, y), the first and second fundamental forms of $(\tilde{M}, \tilde{F})$ are written in the following form:

$$I = -\sin^2\frac{\phi}{2}\,\mathrm{d}x^2 + \cos^2\frac{\phi}{2}\,\mathrm{d}y^2, \qquad II = \sin\frac{\phi}{2}\cos\frac{\phi}{2}(\mathrm{d}x^2 + \mathrm{d}y^2). \tag{5.2}$$

Here ω and ϕ are solutions of Sinh–Laplace and Sine–Laplace equations respectively.

$$\omega_{xx} + \omega_{yy} = \sinh\omega, \qquad \phi_{xx} + \phi_{yy} = \sin\phi.$$

The immersions F and $\tilde{F}$ are related by the following line congruence.

$$\tilde{F} = F + \cosh\theta\left\{\frac{\sin(\phi/2)}{\sinh(\omega/2)}F_x + \frac{\cos(\phi/2)}{\cosh(\omega/2)}F_y\right\}. \tag{5.3}$$

The Sinh–Laplace field ω and Sine–Laplace field ϕ are related by the following formulae:

$$\left(\frac{\phi_x + \omega_y}{2}\right) = -\operatorname{sech}\theta \sinh\frac{\omega}{2}\cos\frac{\phi}{2} - \tanh\theta\cosh\frac{\omega}{2}\sin\frac{\phi}{2},$$
$$\left(\frac{\phi_y - \omega_x}{2}\right) = \operatorname{sech}\theta \cosh\frac{\omega}{2}\sin\frac{\phi}{2} - \tanh\theta\sinh\frac{\omega}{2}\cos\frac{\phi}{2}. \qquad (5.4)$$

With respect to the complex coordinate $z = x + \sqrt{-1}y$, the formulae (5.4) can be written in the following forms:

$$\frac{\partial}{\partial z}(\phi + \sqrt{-1}\omega) = \sqrt{-1}\lambda \sin\left(\frac{\phi - \sqrt{-1}\omega}{2}\right),$$
$$\frac{\partial}{\partial \bar{z}}(\phi - \sqrt{-1}\omega) = -\sqrt{-1}\lambda \sin\left(\frac{\phi - \sqrt{-1}\omega}{2}\right),$$
$$\lambda = -\tanh\theta - \sqrt{-1}\operatorname{sech}\theta \in S^1. \qquad (5.5)$$

The formulae (5.5) are essentially due to G. Leibbrandt [22]. See also [28].

The Bäcklund transformations in Proposition 5.2 induce transformations between harmonic maps from a Riemann surface into H^2 and S^2_1. To describe the induced (Bäcklund) transformations between harmonic maps into H^2 and S^2_1 explicitly is an interesting problem for us. The Darboux form of a Bäcklund transformation on spacelike surfaces with constant curvature will be given in the forthcoming paper [19].

Remark. *The Gauss–Codazzi equations (G) and (C) imply that for every simply connected timelike CMC-1 surface in E^3_1 there exists a timelike extremal surface in S^3_1 which is isometric to the original one (so-called Lawson correspondent.) Using our results on Darboux transformations, we can construct multi-soliton extremal surfaces in S^3_1. Note that timelike extremal surfaces in S^3_1 may be considered as simple mathematical models of rigid strings in particle physics and cosmology* [2,10,33].

Further Sinh–Gordon fields are also related to projective differential geometry. We shall study Darboux transformations on surfaces in real projective 3-space in [20].

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